A Variant of Gödel's Dialectica Interpretation

Bodil Biering (biering@itu.dk)

PLS Group, IT-University of Copenhagen

September 2007 / DOMAINS VIII, Novosibirsk

Brief Overview

- Recall Gödel's Dialectica and the Diller-Nahm variant.
- The "Copenhagen" variant.
- A categorical analysis.
- Future direction.

Gödel's Dialectica (functional) Interpretation

- Recall that Gödel's Dial. translates every formula α of **HA** into a formula of the form $\alpha^D = \exists u \forall x \, \alpha_D(u, x)$, where α_D is a quantifier-free formula of Gödel's system **T**.
- We will focus on the translation of $\wedge$ and $\rightarrow$: Suppose $\alpha^D = \exists u \forall x . \alpha_D(u, x)$ and $\beta^D = \exists v \forall y . \beta_D(v, y)$, then
  - $(\alpha \wedge \beta)^D = \exists u, v \forall x, y . (\alpha_D \wedge \beta_D)$
  - $(\alpha \rightarrow \beta)^D = \exists f : U \Rightarrow V, F : U \times Y \Rightarrow X \forall u, y . (\alpha_D(u, F(u, y)) \rightarrow \beta_D(fu, y))$

HA IS D-INTERPRETED IN T

Gödel's main result was the following:

THEOREM (SOUNDNESS)

*Suppose **HA** $\vdash \alpha$, then there is a term t of **T** such that **T** $\vdash \alpha_D(t, y)$.*

- Soundness is shown by, for each axiom α of **HA** , finding a term (realizer) t of **T** s.t. **T** $\vdash \alpha_D(t, y)$.
- To provide the realizer for the axiom $\alpha \to \alpha \wedge \alpha$, we need atomic formulas to be decidable (recursive).
- The term we want to define is

$$F(u, x, x') = \begin{cases} x & \text{if} \quad \neg\alpha_D(u, x) \\ x' & \text{if} \quad \alpha_D(u, x) \end{cases}$$

The Diller-Nahm variant

- Keep the same interpretation of conjunction, while avoiding the requirement about decidable atomic formulas.
- $$\begin{array}{rcl}(\alpha \to \beta)^{DN} & = & \exists f : U \Rightarrow V, F : U \times Y \Rightarrow \mathbb{P}_f(X).\forall u, y. \\ & & ((\forall x \in F(u,y).\alpha_{DN}(u,x)) \to \beta_{DN}(fu, y))\end{array}$$
- Now the realizer $F : U \times X \times X \to \mathbb{P}_f(X)$ for $(\alpha \to \alpha \wedge \alpha)^{DN}$ is $F(u,x,x') = \{x,x'\}$.

THE COPENHAGEN VARIANT

- Dropping the requirement about decidability results in: the conjunction no longer satisfies $\alpha \to \alpha \wedge \alpha$.
- Change interpretation of conjunction into:

$$(\alpha \wedge \beta)^{D'} = \exists u, v . \forall z \in X + Y . \left(\begin{array}{lll} \texttt{case} & z \in X . & \alpha_{D'}(u, z), \\ \texttt{case} & z \in Y . & \beta_{D'}(v, z) \end{array} \right)$$

- The realizer $F : U \times (X + X) \to X$ is then $F(u, x) = x$
- Note that in the DN case and in the case atomic formulas are decidable, the two definitions of conjunction coincide.

THE COPENHAGEN VARIANT

- In the Copenhagen (Cph) variant conjunction is interpreted as above.
- We find that the corresponding interpretation of implication is:

$$
\begin{array}{ll}
(\alpha \to \beta)^{D'} & = \\
\exists \langle f, F \rangle : & \{U \Rightarrow V \times (U \times Y \Rightarrow X + 1) \mid \\
 & \forall u, y.\mathtt{case}\quad F(u,y) \in X.\alpha_{D'}(u, F(u,y)) \to \beta_{D'}(f(u), y)\} \\
\forall u, y. & \left(\begin{array}{lll} \mathtt{case} & F(u,y) \in 1. & \beta_{D'}(fu, y), \\ \mathtt{case} & F(u,y) \in X. & \top. \end{array} \right)
\end{array}
$$

Soundness for the Copenhagen variant

- Clearly we need a more powerfull type system to express this.
- Let **T**' be Gödel's **T** with the addition of sum (coproduct) types, subset types and universal quantifiers.
- Our categorical analysis shows that if **HA** $\vdash \alpha$ then there is a term t in **T**$'$ s.t. **T**$' \vdash \alpha_{D'}(t, y)$.

Categorical analysis

- Want to analyse the situation where atomic formulas are not decidable.
- Rough idea (by V. de Paiva and Hyland) is to construct category where objects correspond to α^D and a map corresponds to a realizer for $(\alpha \to \beta)^D$.
- Soundness then corresponds to showing that this category (considered as a poset) carries the structure of a Heyting algebra.

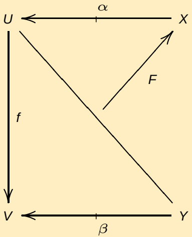
DIALECTICA CATEGORIES

DEFINITION

Let $\mathbb{C}$ *be a category with finite limits, then* $\mathsf{Dial}(\mathbb{C})$ *has*

OBJECTS: *Triples* $A = (U, X, \alpha)$*, where* U, X *are obj. of* $\mathbb{C}$ *and* $\alpha \in \mathsf{Sub}(U \times X)$*.*

MAPS: *A map from* $A = (U, X, \alpha)$ *to* $B = (V, Y, \beta)$ *is a pair of maps* (f, F) *in* $\mathbb{C}$*, such that:*



$$\alpha(u, F(u, y)) \leq \beta(f(u), y) \text{ in } \mathsf{Sub}(U \times Y).$$

THEOREM (VDP)

If $\mathbb{C}$ *is ccc with stable, disjoint coproducts, then* $\mathsf{Dial}(\mathbb{C})$ *has* $1, \times, \otimes, I, \multimap$*.*

Diller-Nahm categories

We have a comonad $! : \mathsf{Dial}(\mathbb{C}) \to \mathsf{Dial}(\mathbb{C})$ defined as follows:

$$!(U, X, \alpha) = (U, X^*, !\alpha)$$

Where X^* is the free commutative monoid, and $!\alpha$ is given by the predicate

$$!\alpha(u, e) \text{ iff } \forall x \in e.\alpha(u, x).$$

The comonad satisfies a distributive law:

$$!(A \times B) \cong !A \otimes !B$$

Definition

The Diller-Nahm category over $\mathbb{C}$ is the Kleisli category $\mathsf{Dial}_!(\mathbb{C})$ for the comonad !.

It is ccc by the following:

$$\begin{aligned} \mathsf{Dial}_!(A \times B, C) &\cong \mathsf{Dial}(!(A \times B), C) \\ &\cong \mathsf{Dial}(!A \otimes !B, C) \\ &\cong \mathsf{Dial}(!A, !B \multimap C) \\ &\cong \mathsf{Dial}_!(A, !B \multimap C) \end{aligned}$$

CATEGORIES FOR THE COPENHAGEN VARIANT

These are also given by a comonad, but it is not distributive so the exponential does not come automatically (as $L^+B \multimap C$).
$L^+ : \mathrm{Dial}(\mathbb{C}) \to \mathrm{Dial}(\mathbb{C})$ is given by

$$L^+(U, X, \alpha) = (U, X+1, \hat{\alpha})$$

where

$$\hat{\alpha}(u, x) = \alpha(u, x)$$
$$\hat{\alpha}(u, *) = \top$$

DEFINITION

The Copenhagen category over $\mathbb{C}$ is the Kleisli category $\mathrm{Dial}^+(\mathbb{C})$ for the comonad L^+.

$\mathrm{Dial}^{+}(\mathbb{C})$ IS WEAKLY CARTESIAN CLOSED

- $\mathrm{Dial}^{+}(\mathbb{C})(A, B)$ is the set of realizers, $f : U \to V, F : U \times Y \to X + 1$ s.t. $\hat{\alpha}(u, F(u, y)) \leq \beta(fu, y)$.
- Under certain conditions on $\mathbb{C}$ we can define a weak exponential, i.e., there is a retract in $\mathrm{Dial}^{+}(\mathbb{C})$:

$$\mathrm{Hom}(A \times B, C) \underset{R}{\overset{I}{\rightleftarrows}} \mathrm{Hom}(A, [B, C]), \qquad RI = \mathrm{Id}$$

  natural in A.
- Using the Cauchy completion, we can obtain a Cartesian closed category.
- Naturality also ensures that the evaluation map behaves nice.

$\mathsf{Dial}^+(\mathbb{C})$ IS WEAKLY CARTESIAN CLOSED

- For the D' interpretation from **HA** into **T**', we only ask for existence of a realizer, this corresponds to taking the preorder reflection of $\mathsf{Dial}^+(\mathbb{C})$ and we get an isomorphism

$$\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, [B, C]).$$

- Our categorical results show furthermore that we can D'-interpret intensional λ-calculus (β-rule, but no η-rule).

Dial^+ is weakly Cartesian closed

We have the following theorem for subobject fibrations:

Theorem

Let $\mathbb{C}$ be a (weakly) ccc with finite limits and stable, disjoint coproducts (i.e. $\mathbb{C}$ is extensive). Suppose we have fibred exponentials ($\rightarrow$ in $\mathsf{Sub}(X)$) and simple products ($\forall$), then $\mathsf{Dial}^+(\mathbb{C})$ is weakly Cartesian closed.

A Dialectica tripos

The Dialectica tripos d is an indexed version of (preordered) Dialectica categories.

Definition

*For $I \in$ **Set**, the fibre over I has objects $(U_i, X_i, \alpha_i)_{i \in I}$, where $0 \in U_i, X_i \subseteq \mathbb{N}$, $\alpha_i \subseteq U_i \times X_i$ and the order is*

$$(U_i, X_i, \alpha_i)_{i \in I} \vdash (V_i, Y_i, \beta_i)$$

iff there exists $f, F \in \mathbb{N}$ s.t.

$$f \in \bigcap_{i \in I} (U_i \Rightarrow V_i) \text{ and } F \in \bigcap_{i \in I} (U_i \times Y_i \Rightarrow X_i)$$

and for all $i \in I, u \in U_i, y \in Y_i$.

$$\alpha_i(u, F(u, y)) \supset \beta_i(fu, y).$$

The fibred exponentials are defined like the Cph variant. The points $0 \in U_i$ are crucial for showing this.

An unnatural retract

Generalizing the idea of having points in the types results in a category with "almost weak exponentials" meaning that there is a retract in Dial($\mathbb{C}$):

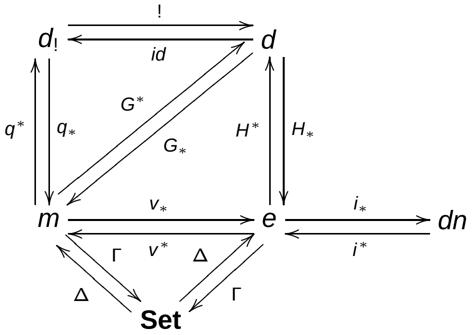
$$\mathrm{Hom}(A \times B, C) \underset{R}{\overset{I}{\rightleftarrows}} \mathrm{Hom}(A, [B, C]), \qquad RI = \mathrm{Id}$$

but it is *not* natural in A.
Notice that this does not matter when we only consider preorders (like in the tripos).

Proposition

There is an isomorphism of triposes $d \cong d^+$.

We have a commuting diagram of indexed adjunctions:



where H, v, and q all are connected geometric morphisms, so they lift to connected geometric morphisms on the induced toposes, and i is a geometric inclusion.