A Variant of Gödel’s Dialectica Interpretation

Bodil Biering (biering@itu.dk)

PLS Group, IT-University of Copenhagen

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Brief Overview

- Recall Gödel’s Dialectica and the Diller-Nahm variant.
- The “Copenhagen” variant.
- A categorical analysis.
- Future direction.
Recall that Gödel’s Dial. translates every formula $\alpha$ of HA into a formula of the form $\alpha^D = \exists u \forall x. \alpha_D(u, x)$, where $\alpha_D$ is a quantifier-free formula of Gödel’s system T.

We will focus on the translation of $\land$ and $\rightarrow$: Suppose $\alpha^D = \exists u \forall x. \alpha_D(u, x)$ and $\beta^D = \exists v \forall y. \beta_D(v, y)$, then

- $(\alpha \land \beta)^D = \exists u, v \forall x, y.(\alpha_D \land \beta_D)$
- $(\alpha \rightarrow \beta)^D = \exists f : U \Rightarrow V, F : U \times Y \Rightarrow X \forall u, y.(\alpha_D(u, F(u, y)) \rightarrow \beta_D(fu, y))$
Gödel’s main result was the following:

**Theorem (Soundness)**

Suppose $\mathsf{HA} \vdash \alpha$, then there is a term $t$ of $\mathsf{T}$ such that $\mathsf{T} \vdash \alpha_D(t, y)$.

- Soundness is shown by, for each axiom $\alpha$ of $\mathsf{HA}$, finding a term (realizer) $t$ of $\mathsf{T}$ s.t. $\mathsf{T} \vdash \alpha_D(t, y)$.
- To provide the realizer for the axiom $\alpha \rightarrow \alpha \land \alpha$, we need atomic formulas to be decidable (recursive).
- The term we want to define is

$$F(u, x, x') = \begin{cases} x & \text{if } \neg \alpha_D(u, x) \\ x' & \text{if } \alpha_D(u, x) \end{cases}$$
The Diller-Nahm variant

- Keep the same interpretation of conjunction, while avoiding the requirement about decidable atomic formulas.

\[(\alpha \rightarrow \beta)^{DN} = \exists f : U \Rightarrow V, F : U \times Y \Rightarrow P_f(X). \forall u, y. ((\forall x \in F(u, y). \alpha_{DN}(u, x)) \rightarrow \beta_{DN}(fu, y))\]

- Now the realizer \( F : U \times X \times X \rightarrow P_f(X) \) for \((\alpha \rightarrow \alpha \land \alpha)^{DN}\) is \(F(u, x, x') = \{x, x'\}\).
Dropping the requirement about decidability results in: the conjunction no longer satisfies $\alpha \rightarrow \alpha \land \alpha$.

Change interpretation of conjunction into:

$$(\alpha \land \beta)^{D'} = \exists u, v. \forall z \in X + Y. \left( \begin{array}{l} \text{case } z \in X. \alpha_{D'}(u, z), \\
\text{case } z \in Y. \beta_{D'}(v, z) \end{array} \right)$$

The realizer $F : U \times (X + X) \rightarrow X$ is then $F(u, x) = x$

Note that in the DN case and in the case atomic formulas are decidable, the two definitions of conjunction coincide.
In the Copenhagen (Cph) variant conjunction is interpreted as above. We find that the corresponding interpretation of implication is:

\[(\alpha \rightarrow \beta)^{D'} = \exists \langle f, F \rangle : \{ U \Rightarrow V \times (U \times Y \Rightarrow X + 1) \mid \forall u, y. \text{case } F(u, y) \in X. \alpha_{D'}(u, F(u, y)) \rightarrow \beta_{D'}(f(u), y) \}\]

\[\forall u, y. \begin{cases} \text{case } F(u, y) \in 1. & \beta_{D'}(fu, y), \\ \text{case } F(u, y) \in X. & \top. \end{cases} \]
Clearly we need a more powerful type system to express this.

Let $T'$ be Gödel’s $T$ with the addition of sum (coproduct) types, subset types and universal quantifiers.

Our categorical analysis shows that if $\text{HA} \vdash \alpha$ then there is a term $t$ in $T'$ s.t. $T' \vdash \alpha_{D'}(t, y)$. 


CATEGORICAL ANALYSIS

- Want to analyse the situation where atomic formulas are not decidable.
- Rough idea (by V. de Paiva and Hyland) is to construct category where objects correspond to $\alpha^D$ and a map corresponds to a realizer for $(\alpha \rightarrow \beta)^D$.
- Soundness then corresponds to showing that this category (considered as a poset) carries the structure of a Heyting algebra.
**Definition**

Let $\mathbb{C}$ be a category with finite limits, then $\text{Dial}(\mathbb{C})$ has

**Objects:** Triples $A = (U, X, \alpha)$, where $U, X$ are obj. of $\mathbb{C}$ and $\alpha \in \text{Sub}(U \times X)$.

**Maps:** A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps $(f, F)$ in $\mathbb{C}$, such that:

\[
\alpha(u, F(u, y)) \leq \beta(f(u), y) \text{ in } \text{Sub}(U \times Y).
\]

**Theorem (VdP)**

*If $\mathbb{C}$ is ccc with stable, disjoint coproducts, then $\text{Dial}(\mathbb{C})$ has $1, \times, \otimes, I, \to$.***
DILLER-NAHM CATEGORIES

We have a comonad $! : \text{Dial}(C) \rightarrow \text{Dial}(C)$ defined as follows:

$!(U, X, \alpha) = (U, X^*, !\alpha)$

Where $X^*$ is the free commutative monoid, and $!\alpha$ is given by the predicate

$!\alpha(u, e)$ iff $\forall x \in e. \alpha(u, x)$.

The comonad satisfies a distributive law:

$!(A \times B) \cong !A \otimes !B$

DEFINITION

The Diller-Nahm category over $C$ is the Kleisli category $\text{Dial}(C)$ for the comonad $!$.

It is ccc by the following:

\begin{align*}
\text{Dial}(A \times B, C) & \quad \text{Dial}(!(A \times B), C) \\
\text{Dial}(!A \otimes !B, C) & \quad \text{Dial}(!A, !B \rightarrow C) \\
\text{Dial}(A, !B \rightarrow C) & \quad \text{Dial}(!A \times B, C)
\end{align*}
CATEGORIES FOR THE COPENHAGEN VARIANT

These are also given by a comonad, but it is not distributive so the exponential does not come automatically (as $L^+B \to C$).

$L^+ : \text{Dial}(C) \to \text{Dial}(C)$ is given by

$$L^+(U, X, \alpha) = (U, X + 1, \hat{\alpha})$$

where

$$\hat{\alpha}(u, x) = \alpha(u, x)$$
$$\hat{\alpha}(u, \ast) = \top$$

**Definition**

The Copenhagen category over $C$ is the Kleisli category $\text{Dial}^+(C)$ for the comonad $L^+$. 
\( \textbf{Dial}^+ (C) \) \textbf{is weakly Cartesian closed}

- \( \textbf{Dial}^+ (C)(A, B) \) is the set of realizers, \( f : U \to V, F : U \times Y \to X + 1 \) s.t. \( \hat{\alpha}(u, F(u, y)) \leq \beta(fu, y) \).
- Under certain conditions on \( C \) we can define a weak exponential, i.e., there is a retract in \( \textbf{Dial}^+ (C) \):

\[
\begin{array}{c}
\text{Hom}(A \times B, C) \\
\text{Hom}(A, [B, C])
\end{array}
\xrightarrow{R} \\
\xleftarrow{I}
\]

natural in \( A \).
- Using the Cauchy completion, we can obtain a Cartesian closed category.
- Naturality also ensures that the evaluation map behaves nice.
For the $D'$ interpretation from $\text{HA}$ into $\text{T'}$, we only ask for existence of a realizer, this corresponds to taking the preorder reflection of $\text{Dial}^+(\mathbb{C})$ and we get an isomorphism

$$\text{Hom}(A \times B, C) \cong \text{Hom}(A, [B, C]).$$

Our categorical results show furthermore that we can $D'$-interpret intensional $\lambda$-calculus ($\beta$-rule, but no $\eta$-rule).
Dial$^+$ is weakly Cartesian closed

We have the following theorem for subobject fibrations:

**Theorem**

Let $\mathcal{C}$ be a (weakly) ccc with finite limits and stable, disjoint coproducts (i.e. $\mathcal{C}$ is extensive). Suppose we have fibred exponentials ($\rightarrow$ in $\text{Sub}(X)$) and simple products ($\forall$), then $\text{Dial}^+(\mathcal{C})$ is weakly Cartesian closed.
The Dialectica tripos $d$ is an indexed version of (preordered) Dialectica categories.

**Definition**

For $I \in \textbf{Set}$, the fibre over $I$ has objects $(U_i, X_i, \alpha_i)_{i \in I}$, where $0 \in U_i$, $X_i \subseteq \mathbb{N}$, $\alpha_i \subseteq U_i \times X_i$ and the order is

$$(U_i, X_i, \alpha_i)_{i \in I} \vdash (V_i, Y_i, \beta_i)$$

iff there exists $f, F \in \mathbb{N}$ s.t.

$$f \in \bigcap_{i \in I} (U_i \Rightarrow V_i) \quad \text{and} \quad F \in \bigcap_{i \in I} (U_i \times Y_i \Rightarrow X_i)$$

and for all $i \in I, u \in U_i, y \in Y_i$.

$$\alpha_i(u, F(u, y)) \supset \beta_i(fu, y).$$

The fibred exponentials are defined like the Cph variant. The points $0 \in U_i$ are crucial for showing this.
AN UNNATURAL RETRACT

Generalizing the idea of having points in the types results in a category with “almost weak exponentials” meaning that there is a retract in Dial(\(\mathbb{C}\)):

\[
\text{Hom}(A \times B, C) \xrightarrow{I} \text{Hom}(A, [B, C]), \quad RI = \text{Id}
\]

but it is not natural in \(A\).

Notice that this does not matter when we only consider preorders (like in the tripos).

**PROPOSITION**

*There is an isomorphism of triposes \(d \simeq d^+\).*
We have a commuting diagram of indexed adjunctions:

where $H$, $\nu$, and $q$ all are connected geometric morphisms, so they lift to connected geometric morphisms on the induced toposes, and $i$ is a geometric inclusion.