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Domain Semantics of Possibility Computations

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Introduction and Background

- An important problem in domain theory is the modeling of non-deterministic feature of programming languages.

- To describe this behaviour, powerdomains were introduced by Plotkin (1976, 1982) and Smyth(1978).

- A classical powerdomain over a domain X is a subset of the power set of X.

- Three classical powerdomain constructions, called the convex, upper, and lower powerdomains, often referred to as Plotkin, Smyth, and Hoare powerdomains.

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Introduction and Background

- Probabilistic non-determinism has also been studied and has led to the probabilistic powerdomain as a model–Saheb(1980) and Jones-Plotkin(1989).
- Different runs of a probabilistic program with the same input may again result in different outputs.

- A probability distribution or continuous valuation on the domain of final states is chosen to describe such a behaviour.
- Jone-Plotkin model of probabilistic computation– the probability is appointed to a subset, which shows that the probability of which a state is in that set.
- Probability distributions on the domain of final states are functions on certain subsets of final states to the unit interval $[0, 1]$.

Introduction and Background

- Which subsets of dcpos are candidates for probabilistic computations— Scott open sets in Jones-Plotkin model.
- The goal of computing is to give, for a given input, the probability distribution on Scott open sets of the domain of final states. — called valuations.
- Valuations are indeed those Scott continuous functions from the Scott topology $\sigma(X)$ to the interval $[0, 1]$, subject to the modularity law:

$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V).$$

- The probabilistic powerdomain $\mathcal{P}(X)$ of a dcpo X as being the family of all such valuations over X ordered pointwise.
- The denotational semantics $[\![C]\!]$ of non-deterministic program C from dcpo D to E, in probability model, is a Scott continuous function from D to the probability powerdomain $\mathcal{P}(E)$.

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Introduction and Background

- In this presentation, we consider a kind of non-deterministic computations, called possibility computations.

- We define a possibility distribution on Scott open sets of the dcpo of states.
- The goal of this kind of non-deterministic computations is to give, for a given input, the possibility distribution on Scott topology of the domain of final states.
- This possibility distributions will justify the axiomatic rule of possibility measures, namely

$$\Pi(U \cup V) = \max(\Pi(U), \Pi(V))$$

where $\Pi(U)$ and $\Pi(V)$ lie in the unit interval $[0, 1]$ and are degrees of possibility of the Scott open set U and V.

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Introduction and Background

- Possibility theory is motivated by the observation that non-determinism can arise through uncertainty, or through unsharpness of data.
- Possibility measure are a concept used in possibility theory. They belong to the broad field of theories of evidence in Artificial Intelligence and Empirical Sciences.(de Coorman and Ruan, 1995).
- Heckmann and Huth (1997,1998) developed an algebraic theory of possibility measures in a general topological setting and investigated the powerdomain of possibility measures.

Introduction and Background

- This presentation will centralize on the domain semantics of possibility computations so as to deal with non-determinism.
- Both denotational and logical semantics in the framework of domain theory are established and their equivalence are verified.
- The denotational semantics–assigning to programs possibility state transformers, i.e., Scott continuous functions from input states to the possibility powerdomain of final states.
- This possibility powerdomain of a dcpo–consisting of all possibility valuations, ordered pointwise, of a dcpo.
- The logical semantics–given by fuzzy predicate transformers [Chen & Jung, 2004].
- Two semantics equivalence will be verified in terms of the integration of fuzzy predicates with respect to possibility valuations.
- The categorical monad of possibility powerdomain –investigated.

Valuations and Denotational Semantics

- Domain basic concepts–dcpo D, (Scott) continuous function, (Scott) open sets, and (Scott) topology $\sigma(D)$.

- Possibility valuations give, for every open sets (or property), the possibility that the result of a possibility computation is in this set (or satisfies this property).

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Possibility Valuations

Definition Let D be a dcpo. A function $\Pi : \sigma(D) \longrightarrow [0,1]$ is called a possibility valuation of D, if Π satisfies the following conditions:

1 Strictness. $\Pi(\emptyset) = 0$;

2 Monotonicity. $V \subseteq U$ implies $\Pi(V) \leq \Pi(U)$;

3 Max-modularity. $\Pi(U \cup V) = \Pi(U) \vee \Pi(V)$; and

4 Continuity. If $\{U_i : i \in I\}$ is any directed subset of $\sigma(D)$, then $\Pi(\bigcup_{i \in I} U_i) = \sup_{i \in I} \Pi(U_i)$.

We denote the collection of all possibility valuations of D by $\pi(D)$, which will be called the possibility powerdomain of D, being ordered by the pointwise order $\sqsubseteq$, i.e., $\Pi \sqsubseteq \Pi'$ iff $\forall U \in \sigma(D).\Pi(U) \leq \Pi'(U)$, and $(D, \sigma(D), \Pi)$ will be said to be a possibility valuation space.

Remark: Possibility valuations = Huth's possibility measures.

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Denotational Semantics

Definition Let D and E be dcpos. The denotational semantics of a possibility computation F from D to E is assigned to a Scott continuous function $[\![F]\!] : D \longrightarrow \pi(E)$.

Remark: $\pi(\mathbf{1}) \cong [0,1]$.

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Fuzzy Predicate Transformers

- Classic triple:

$$\{x >= 10\}x := x + 1\{x >= 6\}.$$

We notice that $\mathrm{wp}(x := x + 1, \{x >= 6\}) = \{x >= 5\}$.

- Fuzzy triple:

$$\{x \text{ is approximative to } 10\}x := x + 1\{x \text{ is approximative to } 6\}.$$

We might think that

$$\begin{aligned} & \mathrm{wp}(x := x + 1, \{x \text{ is approximative to } 6\}) \\ = \; & \{x \text{ is approximative to } 5\}. \end{aligned}$$

Fuzzy Predicate Transformers

- **Definition :** Let D be a dcpo. Scott-continuous functions from D into $[0,1]$ are called fuzzy predicates. The set of all fuzzy predicates on D is denoted as $\mathcal{F}(D)$.
- **Decomposition Theorem:** Let f be a fuzzy predicate on a dcpo D. Then

$$f = \bigvee_{r\in[0,1)} (r \wedge \chi_{f^{-1}(r,1]}).$$

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Fuzzy Predicate Transformers

- A fuzzy predicate transformer t from dcpo D to dcpo E is a mapping from $\mathcal{F}(E)$ to $\mathcal{F}(D)$ in a backward way, i.e.,

$$\mathrm{t} : \mathcal{F}(E) \longrightarrow \mathcal{F}(D).$$

- **Definition:** A fuzzy predicate transformer t from dcpo D to dcpo E is said to be healthy, if it satisfies the following healthy conditions:

(1) Sups-preserving : If $\{f_i : i \in I\} \subseteq \mathcal{F}(D)$, then $t(\bigvee_{i\in I} f_i) = \bigvee_{i\in I} t(f_i)$, i.e., t preserves arbitrary sups.

(2) Level-preserving: For any $r \in [0,1]$ and $U \in \sigma(D)$, $t(r \wedge \chi_U) = r \wedge t(\chi_U)$ holds.

The notation $[\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$ will denote the set of all healthy fuzzy predicate transformers from dcpo D to dcpo E with the pointwise order.

Logical Semantics

- **Definition:** Let D and E be dcpos. The logical semantics of a possibility computation F from D to E is assigned to a healthy fuzzy predicate transformer from E to D. This logical semantics will be denoted as $\|F\|$.

- A logical semantics can induce the denotational semantics of a possibility computation.

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Logical Semantics

- We define a mapping

$$\alpha : [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)] \longrightarrow [D \longrightarrow \pi(E)]$$

by setting:

$$\alpha(t)(x)(U) = t(\chi_U)(x)$$

for any $t \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)], x \in D$, and $U \in \sigma(E)$.

- **Theorem** For any $t \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$ and $x \in \mathrm{D}$, $\Pi = \alpha(t)(x)$ is a possibility valuation of E, i.e., $\alpha(t)(x) \in \pi(E)$.

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The Proof

Theorem For any $t \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$ and $x \in$D, $\Pi = \alpha(t)(x)$ is a possibility valuation of E, i.e., $\alpha(t)(x) \in \pi(E)$.

- Since $\alpha(t)(x)(\emptyset) = t(\chi_\emptyset)(x) = t(0 \wedge \chi_\emptyset)(x) = 0 \wedge t(\chi_\emptyset)(x) = 0$, $\alpha(t)(x)$ is strict.
- For any $\{U_i : i \in I\} \subseteq \sigma(E)$,

$$\begin{aligned} \alpha(t)(x)(\bigcup_{i\in I} U_i) &= t(\chi_{\cup_{i\in I} U_i})(x) \\ &= t(\bigvee_{i\in I} \chi_{U_i})(x) \\ &= (\bigvee_{i\in I} t(\chi_{U_i}))(x) \quad (t \text{ preserves arbitrary sups}) \\ &= \sup_{i\in I} \alpha(t)(x)(U_i). \end{aligned}$$

  So, $\alpha(t)(x)$ preserves arbitrary sups.
- That is, $\alpha(t)(x)$ is a possibility valuation over dcpo E.

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Logical Semantics

- **Theorem** For any $t \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$, $\alpha(t)$ is Scott continuous from D to $\pi(E)$, i.e., $\alpha(t) \in [D \longrightarrow \pi(E)]$.

- We know that for a possibility computation C, if we have gotten its fuzzy logical semantics $\|C\|$, then we can get the corresponding denotational semantics $[\![C]\!] = \alpha(\|F\|)$.

Fuzzy Integration and Equivalence Between Semantics

- We define the integration of a fuzzy predicate f of a dcpo D over a possibility valuation space $(D, \sigma(D), \Pi)$ as follows:
- **Definition** Let D be a dcpo, f a fuzzy predicate of D, and Π a possibility valuation of D. The integral of f with respect to Π over a Scott open set U of D is defined as

$$\int_U f d\Pi = \sup_{\alpha \in [0,1]} [\alpha \wedge \Pi(f^{-1}(\alpha, 1] \cap U)].$$

- Particularly, $\int_D f d\Pi = \sup_{\alpha \in [0,1]} [\alpha \wedge \Pi(f^{-1}(\alpha, 1])]$.
- We write $\int f \, d\Pi$ for $\int_D f d\Pi$.

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Fuzzy Integration and Equivalence Between Semantics

- Fuzzy Integration can be used to define a mapping β as follows:

$$\beta : [D \longrightarrow \pi(E)] \longrightarrow [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$$

by setting: for any $h \in [D \longrightarrow \pi(E)]$, $f \in \mathcal{F}(E)$ and $x \in D$

$$\beta(h)(f)(x) = \int f \, dh(x).$$

- **Theorem** $\beta(h) \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$, for any $h \in [D \longrightarrow \pi(E)]$.

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Fuzzy Integration and Equivalence Between Semantics

- The main result:
  - For all $h \in [D \longrightarrow \pi(E)]$ and $t \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$, we have $\alpha(\beta(h)) = h$ and $\beta(\alpha(t)) = t$.
  - $[\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)] \cong [D \longrightarrow \pi(E)]$, by α for the implication $\longrightarrow$ and β for another implication $\longleftarrow$.

Monad of Possibility Powerdomain

A monad in a category **C** is a triple (T, η, μ) consisting of an endfunctor $T : \mathbf{C} \to \mathbf{C}$ and two natural transformation

$$\eta : id_{\mathbf{C}} \to T, \quad \mu : T^2 \to T,$$

which satisfy the following equations:

$$\mu \circ T\eta = \mu \circ \eta T = id_T \ \text{ and } \ \mu \circ T\mu = \mu \circ \mu T.$$

That is, the following two diagrams are commutative:

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "id_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "id_T"] \\
& T &
\end{tikzcd}$$

and

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

Monad of Possibility Powerdomain

Claim: An operation m on the objects of a category is part of a monadic functor $T : \mathbf{C} \longrightarrow \mathbf{C}$ iff there exist an operation † which takes a morphism $f : X \to m(Y)$ to $f^\dagger : m(X) \to m(Y)$ and a morphism $i_X : X \to m(X)$ which obey equations below

$$(i_X)^\dagger = id_{m(X)}$$

$$f^\dagger \circ i_X = f$$

$$(g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger.$$

The functor T is defined by $T(X) = m(X)$ and $T(f) = (i_Y \circ f)^\dagger$ and two natural transformers η and μ are respectively obtained by

$$\eta_X = i_X \text{ and } \mu_X = id^\dagger_{m(X)}.$$

Monad of Possibility Powerdomain

- Define the operator †.
  for a given $f : X \to \pi(Y)$ and a $\Pi \in \pi(X)$,
  $$f^\dagger(\Pi)(O) = \int_{x \in X} f(x)(O)\,d\Pi \tag{1}$$
  for any $O \in \sigma(Y)$.
- Define the operation i by, for any dcpo X
  $$\begin{aligned} i_X : \; X & \to \pi(X) \\ x & \longmapsto \theta_x \end{aligned}$$
  where θ_x is defined by, for any $O \in \sigma(D)$,
  $$\theta_x(O) = \begin{cases} 1 & \text{if } x \in O \\ 0 & \text{otherwise.} \end{cases}$$

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Monad of Possibility Powerdomain

Theorem The possibility powerdomain operator π is a part of a monadic functor of the category **Dcpo.**

Conclusion and Future Consideration

- Proposed a kind of computational model, called possibility computations, providing a new approach to deal with the non-deterministic computing.
- Possibility valuations takes as the serving of denotational semantics of non-deterministic computations, and fuzzy predicate transformers as logical semantics.
- We proved the equivalence between these two semantics.
- Gave the monadness of the possibility powerdomain operators in the category of dcpos and Scott continuous functions.
- How to set up the semantics for an abstract programming language needs to consider furthermore.
- We will pay attention to how to establish the semantical model of computations in which possibility and non-determinism coexist.
- The similar consideration for probability case has been considered by He-Seidel-McIver(1997), Tix-Keimel-Plotkin(2005) and Ying(2003).

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