Domain Semantics of Possibility Computations

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Introduction and Background

• An important problem in domain theory is the modeling of nondeterministic feature of programming languages.

• To describe this behaviour, powerdomains were introduced by Plotkin (1976, 1982) and Smyth (1978).

• A classical powerdomain over a domain $X$ is a subset of the power set of $X$.

• Three classical powerdomain constructions, called the convex, upper, and lower powerdomains, often referred to as Plotkin, Smyth, and Hoare powerdomains.
Introduction and Background

- Probabilistic non-determinism has also been studied and has led to the probabilistic powerdomain as a model—Saheb (1980) and Jones-Plotkin (1989).

- Different runs of a probabilistic program with the same input may again result in different outputs.

- A probability distribution or continuous valuation on the domain of final states is chosen to describe such a behaviour.

- Jone-Plotkin model of probabilistic computation— the probability is appointed to a subset, which shows that the probability of which a state is in that set.

- Probability distributions on the domain of final states are functions on certain subsets of final states to the unit interval $[0, 1]$. 
Introduction and Background

• Which subsets of dcpoś are candidates for probabilistic computations—Scott open sets in Jones-Plotkin model.

• The goal of computing is to give, for a given input, the probability distribution on Scott open sets of the domain of final states. —called valuations.

• Valuations are indeed those Scott continuous functions from the Scott topology $\sigma(X)$ to the interval $[0,1]$, subject to the modularity law:

$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V).$$

• The probabilistic powerdomain $\mathcal{P}(X)$ of a dcpo $X$ as being the family of all such valuations over $X$ ordered pointwise.

• The denotational semantics $[C]$ of non-deterministic program $C$ from dcpo $D$ to $E$, in probability model, is a Scott continuous function from $D$ to the probability powerdomain $\mathcal{P}(E)$. 
In this presentation, we consider a kind of non-deterministic computations, called **possibility computations**.

We define a **possibility distribution** on Scott open sets of the dcpo of states.

The **goal** of this kind of non-deterministic computations is to give, for a given input, the **possibility distribution** on Scott topology of the domain of final states.

This possibility distributions will justify the **axiomatic** rule of possibility measures, namely

\[ \Pi(U \cup V) = \max(\Pi(U), \Pi(V)) \]

where \( \Pi(U) \) and \( \Pi(V) \) lie in the unit interval \([0, 1]\) and are degrees of possibility of the Scott open set \( U \) and \( V \).
Introduction and Background

- **Possibility theory** is motivated by the observation that non-determinism can arise through uncertainty, or through unsharpness of data.

- **Possibility measure** are a concept used in possibility theory. They belong to the broad field of theories of evidence in Artificial Intelligence and Empirical Sciences. (de Cooman and Ruan, 1995).

- Heckmann and Huth (1997, 1998) developed an algebraic theory of possibility measures in a general topological setting and investigated the powerdomain of possibility measures.
Introduction and Background

- This presentation will centralize on the **domain semantics** of possibility computations so as to deal with non-determinism.
- Both **denotational and logical** semantics in the framework of domain theory are established and their **equivalence** are verified.
- The denotational semantics—assigning to programs possibility state transformers, i.e., Scott continuous functions from input states to the possibility powerdomain of final states.
- This **possibility powerdomain** of a dcpo—consisting of all possibility valuations, ordered pointwise, of a dcpo.
- The logical semantics—given by **fuzzy predicate transformers** [Chen & Jung, 2004].
- Two semantics equivalence will be verified in terms of the **integration** of fuzzy predicates with respect to possibility valuations.
- The **categorical monad** of possibility powerdomain –investigated.
Valuations and Denotational Semantics

• Domain basic concepts—dcpo $D$, (Scott) continuous function, (Scott) open sets, and (Scott) topology $\sigma(D)$.

• Possibility valuations give, for every open sets (or property), the possibility that the result of a possibility computation is in this set (or satisfies this property).
Possibility Valuations

**Definition** Let $D$ be a dcpo. A function $\Pi : \sigma(D) \rightarrow [0, 1]$ is called a possibility valuation of $D$, if $\Pi$ satisfies the following conditions:

1. **Strictness.** $\Pi(\emptyset) = 0$;
2. **Monotonicity.** $V \subseteq U$ implies $\Pi(V) \leq \Pi(U)$;
3. **Max-modularity.** $\Pi(U \cup V) = \Pi(U) \lor \Pi(V)$; and
4. **Continuity.** If $\{U_i : i \in I\}$ is any directed subset of $\sigma(D)$, then $\Pi(\bigcup_{i \in I} U_i) = \sup_{i \in I} \Pi(U_i)$.

We denote the collection of all possibility valuations of $D$ by $\pi(D)$, which will be called the possibility powerdomain of $D$, being ordered by the pointwise order $\sqsubseteq$, i.e., $\Pi \sqsubseteq \Pi'$ iff $\forall U \in \sigma(D).\Pi(U) \leq \Pi'(U)$, and $(D, \sigma(D), \Pi)$ will be said to be a possibility valuation space.

**Remark:** Possibility valuations = Huth’s possibility measures.
Denotational Semantics

**Definition** Let $D$ and $E$ be dcpos. The denotational semantics of a possibility computation $F$ from $D$ to $E$ is assigned to a Scott continuous function $[F] : D \longrightarrow \pi(E)$.

**Remark:** $\pi(1) \cong [0, 1]$. 
Fuzzy Predicate Transformers

- Classic triple:

\[
\{ x \geq 10 \} x := x + 1 \{ x \geq 6 \}.
\]

We notice that \( \text{wp}(x := x + 1, \{ x \geq 6 \}) = \{ x \geq 5 \} \).

- Fuzzy triple:

\[
\{ x \text{ is approximative to } 10 \} x := x + 1 \{ x \text{ is approximative to } 6 \}.
\]

We might think that

\[
\text{wp}(x := x + 1, \{ x \text{ is approximative to } 6 \}) = \{ x \text{ is approximative to } 5 \}.
\]
Fuzzy Predicate Transformers

- **Definition**: Let $D$ be a dcpo. Scott-continuous functions from $D$ into $[0, 1]$ are called **fuzzy predicates**. The set of all fuzzy predicates on $D$ is denoted as $\mathcal{F}(D)$.

- **Decomposition Theorem**: Let $f$ be a fuzzy predicate on a dcpo $D$. Then

$$f = \bigvee_{r \in [0, 1)} (r \land \chi_{f^{-1}(r, 1]}).$$
A fuzzy predicate transformer $t$ from dcpo $D$ to dcpo $E$ is a mapping from $\mathcal{F}(E)$ to $\mathcal{F}(D)$ in a backward way, i.e.,

$$t : \mathcal{F}(E) \to \mathcal{F}(D).$$

**Definition:** A fuzzy predicate transformer $t$ from dcpo $D$ to dcpo $E$ is said to be healthy, if it satisfies the following healthy conditions:

1. **Sups-preserving:** If $\{f_i : i \in I\} \subseteq \mathcal{F}(D)$, then $t(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} t(f_i)$, i.e., $t$ preserves arbitrary sups.
2. **Level-preserving:** For any $r \in [0, 1]$ and $U \in \sigma(D)$, $t(r \wedge \chi_U) = r \wedge t(\chi_U)$ holds.

The notation $[\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$ will denote the set of all healthy fuzzy predicate transformers from dcpo $D$ to dcpo $E$ with the pointwise order.
Logical Semantics

- **Definition:** Let $D$ and $E$ be dcpos. The logical semantics of a possibility computation $F$ from $D$ to $E$ is assigned to a healthy fuzzy predicate transformer from $E$ to $D$. This logical semantics will be denoted as $\|F\|$.

- A logical semantics can *induce* the denotational semantics of a possibility computation.
• We define a mapping

\[ \alpha : [\mathcal{F}(E) \xrightarrow{H} \mathcal{F}(D)] \rightarrow [D \xrightarrow{}\pi(E)] \]

by setting:

\[ \alpha(t)(x)(U) = t(\chi_U)(x) \]

for any \( t \in [\mathcal{F}(E) \xrightarrow{H} \mathcal{F}(D)], x \in D, \text{ and } U \in \sigma(E). \)

• **Theorem**  For any \( t \in [\mathcal{F}(E) \xrightarrow{H} \mathcal{F}(D)] \) and \( x \in D, \) \( \Pi = \alpha(t)(x) \) is a possibility valuation of \( E, \) i.e., \( \alpha(t)(x) \in \pi(E). \)
The Proof

Theorem For any $t \in [\mathcal{F}(E) \longrightarrow_H \mathcal{F}(D)]$ and $x \in D$, $\Pi = \alpha(t)(x)$ is a possibility valuation of $E$, i.e., $\alpha(t)(x) \in \pi(E)$.

- Since $\alpha(t)(x)(\emptyset) = t(\chi_\emptyset)(x) = t(0 \land \chi_\emptyset)(x) = 0 \land t(\chi_\emptyset)(x) = 0$, $\alpha(t)(x)$ is strict.

- For any $\{U_i : i \in I\} \subseteq \sigma(E)$,

\[
\alpha(t)(x)(\bigcup_{i \in I} U_i) = t(\chi_{\bigcup_{i \in I} U_i})(x)
\]

\[
= t(\bigvee_{i \in I} \chi_{U_i})(x)
\]

\[
= (\bigvee_{i \in I} t(\chi_{U_i}))(x) \quad (t \text{ preserves arbitrary sups})
\]

\[
= \sup_{i \in I} \alpha(t)(x)(U_i).
\]

So, $\alpha(t)(x)$ preserves arbitrary sups.

- That is, $\alpha(t)(x)$ is a possibility valuation over dcpo $E$. 

Logical Semantics

- **Theorem** For any \( t \in [\mathcal{F}(E) \rightarrow_H \mathcal{F}(D)] \), \( \alpha(t) \) is Scott continuous from \( D \) to \( \pi(E) \), i.e., \( \alpha(t) \in [D \rightarrow \pi(E)] \).

- We know that for a possibility computation \( C \), if we have gotten its fuzzy logical semantics \( \|C\| \), then we can get the corresponding denotational semantics \( \llbracket C \rrbracket = \alpha(\|F\|) \).
Fuzzy Integration and Equivalence Between Semantics

- We define the integration of a fuzzy predicate $f$ of a dcpo $D$ over a possibility valuation space $(D, \sigma(D), \Pi)$ as follows:

- **Definition** Let $D$ be a dcpo, $f$ a fuzzy predicate of $D$, and $\Pi$ a possibility valuation of $D$. The integral of $f$ with respect to $\Pi$ over a Scott open set $U$ of $D$ is defined as

$$\int_U f \, d\Pi = \sup_{\alpha \in [0, 1]} [\alpha \land \Pi(f^{-1}(\alpha, 1] \cap U)].$$

- Particularly, $\int_D f \, d\Pi = \sup_{\alpha \in [0, 1]} [\alpha \land \Pi(f^{-1}(\alpha, 1))].$

- We write $\int f \, d\Pi$ for $\int_D f \, d\Pi$. 
Fuzzy Integration and Equivalence Between Semantics

- Fuzzy Integration can be used to define a mapping $\beta$ as follows:

\[
\beta : [D \rightarrow \pi(E)] \rightarrow [\mathcal{F}(E) \rightarrow_H \mathcal{F}(D)]
\]

by setting: for any $h \in [D \rightarrow \pi(E)]$, $f \in \mathcal{F}(E)$ and $x \in D$

\[
\beta(h)(f)(x) = \int f \, dh(x).
\]

- Theorem $\beta(h) \in [\mathcal{F}(E) \rightarrow_H \mathcal{F}(D)]$, for any $h \in [D \rightarrow \pi(E)]$. 
Fuzzy Integration and Equivalence Between Semantics

- The main result:

  – For all $h \in [D \rightarrow \pi(E)]$ and $t \in [\mathcal{F}(E) \rightarrow_{H} \mathcal{F}(D)]$, we have $\alpha(\beta(h)) = h$ and $\beta(\alpha(t)) = t$.

  – $[\mathcal{F}(E) \rightarrow_{H} \mathcal{F}(D)] \cong [D \rightarrow \pi(E)]$, by $\alpha$ for the implication $\rightarrow$ and $\beta$ for another implication $\leftarrow$. 
Monad of Possibility Powerdomain

A **monad** in a category $\mathbf{C}$ is a triple $(T, \eta, \mu)$ consisting of an endfunctor $T : \mathbf{C} \to \mathbf{C}$ and two *natural transformation*

$$\eta : id_{\mathbf{C}} \to T, \quad \mu : T^2 \to T,$$

which satisfy the following equations:

$$\mu \circ T \eta = \mu \circ \eta T = id_T \quad \text{and} \quad \mu \circ T \mu = \mu \circ \mu T.$$

That is, the following two diagrams are *commutative*:

\[
\begin{array}{ccc}
T & \xrightarrow{T \eta} & T^2 \\
\downarrow{id_T} & & \downarrow{\mu} \\
T & \xrightarrow{T} & T
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
T^2 & \xrightarrow{\eta T} & T \\
\downarrow{id_T} & & \downarrow{\mu} \\
T & \xrightarrow{\mu} & T
\end{array}
\]

and

\[
\begin{array}{ccc}
T^3 & \xrightarrow{T \mu} & T^2 \\
\downarrow{\mu T} & & \downarrow{\mu} \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\]
Monad of Possibility Powerdomain

Claim: An operation \( m \) on the objects of a category is part of a monadic functor \( T : \mathcal{C} \rightarrow \mathcal{C} \) iff there exist an operation \( \dagger \) which takes a morphism \( f : X \rightarrow m(Y) \) to \( f^\dagger : m(X) \rightarrow m(Y) \) and a morphism \( i_X : X \rightarrow m(X) \) which obey equations below

\[
(i_X)^\dagger = \text{id}_{m(X)}
\]

\[
f^\dagger \circ i_X = f
\]

\[
(g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger.
\]

The functor \( T \) is defined by \( T(X) = m(X) \) and \( T(f) = (i_Y \circ f)^\dagger \) and two natural transformers \( \eta \) and \( \mu \) are respectively obtained by

\[
\eta_X = i_X \text{ and } \mu_X = \text{id}^\dagger_{m(X)}.
\]
Monad of Possibility Powerdomain

- Define the operator \( f^\dagger \).
  for a given \( f : X \rightarrow \pi(Y) \) and a \( \Pi \in \pi(X) \),
  \[
  f^\dagger(\Pi)(O) = \int_{x \in X} f(x)(O) d\Pi
  \]
  for any \( O \in \sigma(Y) \).
- Define the operation \( i \) by, for any dcpo \( X \)
  \[
  i_X : X \rightarrow \pi(X)
  \]
  \[
  x \mapsto \theta_x
  \]
  where \( \theta_x \) is defined by, for any \( O \in \sigma(D) \),
  \[
  \theta_x(O) = \begin{cases} 
  1 & \text{if } x \in O \\
  0 & \text{otherwise.}
  \end{cases}
  \]
Monad of Possibility Powerdomain

Theorem  The possibility powerdomain operator $\pi$ is a part of a monadic functor of the category $\textbf{Dcpo}$. 
Conclusion and Future Consideration

- Proposed a kind of computational model, called possibility computations, providing a new approach to deal with the non-deterministic computing.

- Possibility valuations takes as the serving of denotational semantics of non-deterministic computations, and fuzzy predicate transformers as logical semantics.

- We proved the equivalence between these two semantics.

- Gave the monadness of the possibility powerdomain operators in the category of dcpo's and Scott continuous functions.

- How to set up the semantics for an abstract programming language needs to consider furthermore.

- We will pay attention to how to establish the semantical model of computations in which possibility and non-determinism coexist.

- The similar consideration for probability case has been considered by He-Seidel-McIver(1997), Tix-Keimel-Plotkin(2005) and Ying(2003).
Thanks

Thanks!

Xie Xie!