Bisimilarity, behavioral and logical equivalence for stochastic right coalgebras

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We will discuss the logic given through

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \varphi$$

with $a \in \mathfrak{Act}$, $\mathfrak{Act}$ a countable alphabet of actions.

INTERPRETATION THROUGH KRIPKE MODELS

If $\mathcal{M} = \left(S, (\rightarrow_a)_{a \in \mathfrak{Act}}\right)$ is a Kripke model (a.k.a. labeled transition system) over state space S, then

$$\mathcal{M}, s \models \langle a \rangle \varphi \text{ iff } \exists s' \in [\![\varphi]\!] : s \rightarrow_a s'$$

(here $[\![\varphi]\!] := \{s \in S \mid \mathcal{M}, s \models \varphi\}$, as usual).

THEORY OF A STATE

The theory of state $s \in S$ is as usual the set of formulas which are valid in this state,

$$Th_{\mathcal{M}}(s) := \{\varphi \mid \mathcal{M}, s \models \varphi\}.$$

BISIMILARITY

MILNER'S DEFINITION

$B \subseteq S \times S'$ is a bisimulation between the Kripke models $\mathcal{M}$ and $\mathcal{M}'$ iff for all $\langle s, s' \rangle \in B$

- if $s \to_a t$, then $\exists t' \in S' : s' \to'_a t'$ and $\langle t, t' \rangle \in B$.
- if $s' \to'_a t'$, then $\exists t \in S : s \to_a t$ and $\langle t, t' \rangle \in B$.

ACZEL'S THEOREM

B is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ iff there exists a transition structure $(\to''_a)_{a \in \mathfrak{Act}}$ on B such that

$$\mathcal{M} \leftarrow \left(B, (\to''_a)_{a \in \mathfrak{Act}}\right) \to \mathcal{M}'.$$

Jan Rutten's survey paper on coalgebras, TCS, 2000

Overview

LOGIC

Stochastic version adds a lower bound for the probability of satisfaction to the modal operators:

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_q \varphi$$

with $q \in \mathbb{Q}$ and $a \in \mathfrak{Act}$.

Intuition: State s satisfies $\langle a \rangle_q \varphi$, provided the transition from s upon action a leads to a state that satisfies φ with probability at least q.

STOCHASTIC KRIPKE MODEL

$\mathcal{K} = (S, (k_a)_{a \in \mathfrak{Act}})$ with $k_a(s)(C)$ as the probability that action a in state s leads to a state in the measurable set $C \subseteq S$. Note that $k_a(s)(S) \leq 1$.

INTERPRETATION

$\mathcal{K}, s \models \langle a \rangle_q \varphi$ iff $k_a(s)([\![\varphi]\!]) \geq q$.

MORPHISMS

Let $\mathcal{K} = (S, (k_a)_{a \in \mathfrak{Act}})$ and $\mathcal{K}' = (S', (k'_a)_{a \in \mathfrak{Act}})$ be stochastic Kripke models.

MORPHISM

A measurable and surjective map $f : S \to S'$ is a morphism $\mathcal{K} \to \mathcal{K}'$ iff

$$k'_a(f(s))(B') = k_a(s)\left(f^{-1}\left[B'\right]\right)$$

for every measurable subset $B' \subseteq S'$ and every $s \in S$ and each action $a \in \mathfrak{Act}$.

REMARK

$\mathcal{K}, s \models \varphi \iff \mathcal{K}', f(s) \models \varphi$ for morphism $f : \mathcal{K} \to \mathcal{K}'$.

$\mathcal{K}$ and $\mathcal{K}'$ are

- bisimilar iff $\mathcal{K} \leftarrow \mathcal{K}'' \rightarrow \mathcal{K}'$ for some $\mathcal{K}''$,
- behavioral equivalent iff $\mathcal{K} \rightarrow \mathcal{K}'' \leftarrow \mathcal{K}'$ for some $\mathcal{K}''$,
- logical equivalent iff $\left\{Th_{\mathcal{K}}(s) \mid s \in S\right\} = \left\{Th_{\mathcal{K}'}(s') \mid s' \in S'\right\}$.

JAP-Theorem

Theorem

For stochastic Kripke models over analytic spaces with Borel measurable transition laws, these equivalences hold

$$\text{Logical equivalence} \Leftrightarrow \text{Bisimilarity} \Leftrightarrow \text{Behavioral equivalence.}$$

Remarks

1. Discrete version by K. Larsen and A. Skou, Inf. Comput., 1991.
2. First version in a restricted non-discrete setting established by **J**. Desharnais, **A**. Edalat, and **P**. Panangaden, LICS 1997.
3. General version for general modal logics (SIAM J. Computing 2006) and for continuous time stochastic logics (J. Appl. Logic 2007).

More general principle?

Coalgebraic view

A Markov transition system $\mathcal{K} = (S, (k_a)_{a \in \mathfrak{Act}})$ can be understood as a map

$$k : S \to \prod_{a \in \mathfrak{Act}} \mathbb{S}(S) = (\mathbb{F} \circ \mathbb{S})(S)$$

with $\mathbb{S}(S)$ as the set of all subprobabilities on S.
Thus $\mathcal{K}$ is a coalgebra (S, k) for the functor $\mathbb{F} \circ \mathbb{S}$.

$\models$ in terms of k?

Spell out

$$\begin{aligned} s \models \langle a \rangle_q \varphi &\Leftrightarrow k_a(s)([\![\varphi]\!]) \geq q \\ &\Leftrightarrow k_a(s) \in \{\mu \in \mathbb{S}(S) \mid \mu([\![\varphi]\!]) \geq q\} \\ &\Leftrightarrow s \in k^{-1}\left[\{m \in (\mathbb{F} \circ \mathbb{S})(S) \mid \pi_a(m)([\![\varphi]\!]) \geq q\}\right] \\ &\Leftrightarrow s \in \left(k^{-1} \circ \lambda_S^{a,q}\right)([\![\varphi]\!]) \end{aligned}$$

with $\lambda_S^{a,q}(D) := \{m \in (\mathbb{F} \circ \mathbb{S})(S) \mid \pi_a(m)(D) \geq q\}$.

STRUCTURE OF λ

$\lambda_S^{a,q} : \Big(\text{measurable sets in } S\Big) \to \Big(\text{ measurable sets in } (\mathbb{F} \circ \mathbb{S})(S)\Big)$.
$\lambda^{a,q}$ is a natural transformation, called a predicate lifting for $\mathbb{F} \circ \mathbb{S}$.

Predicate liftings arose originally from the work of L. Moss, D. Pattinson, and L. Schröder.

COALGEBRAIC LOGIC

Formulas are defined through

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \langle \lambda \rangle \varphi$$

with $\lambda \in \Lambda$, Λ a set of predicate liftings for $\mathbb{F} \circ \mathbb{S}$.

STOCHASTIC RIGHT COALGEBRA

If $\mathbb{F}$ is a functor on a suitable category of measurable spaces, then (S, k) is a stochastic right coalgebra for $\mathbb{F}$ iff $k : S \to (\mathbb{F} \circ \mathbb{S})(S)$ is a measurable map, the system dynamics.

MORPHISMS

A measurable map $f : S \to S'$ is a morphism $\mathcal{R} \to \mathcal{R}'$ for the stochastic right coalgebras $\mathcal{R}$ and $\mathcal{R}'$ iff

$$k' \circ f = (\mathbb{F} \circ \mathbb{S})(f) \circ k.$$

$\models$

Interpret $\langle\lambda\rangle\varphi$ in the stochastic right coalgebra $\mathcal{R} = (S, k)$ through

$$\mathcal{R}, s \models \langle\lambda\rangle\varphi \text{ iff } s \in \left(k^{-1} \circ \lambda_S\right)([\![\varphi]\!]).$$

OBSERVATION

Since each $\lambda \in \Lambda$ is natural, we have for the morphism $f : \mathcal{R} \to \mathcal{R}'$

$$\mathcal{R}, s \models \varphi \Leftrightarrow \mathcal{R}', f(s) \models \varphi.$$

Separation issues

The set Λ is assumed to be separating ("there are enough liftings"). Formally:

Separation property

If $Th_{\mathcal{R}}(s) \neq Th_{\mathcal{R}'}(s')$ for states s and s' in a right coalgebra $\mathcal{R}$ resp. $\mathcal{R}'$, then there exists a formula φ and a lifting $\lambda \in \Lambda$ such that

- either $\mathcal{R}, s \models \langle\lambda\rangle\varphi$
- or $\mathcal{R}', s' \models \langle\lambda\rangle\varphi$

holds.

Technically, separation relates

- the equivalence relation ϱ defined by the set of formulas (through $s_1 \,\varrho\, s_2$ iff $\forall\varphi : s_1 \models \varphi \Leftrightarrow s_1 \models \varphi$)
- to the kernel $\ker((\mathbb{F} \circ \mathbb{S})(\eta_\varrho))$ of the image of its factor map η_ϱ under $\mathbb{F} \circ \mathbb{S}$.

GENERALIZATION OF THE JAP-THEOREM

THEOREM

The following holds for stochastic right coalgebras $\mathcal{R}$ and $\mathcal{R}'$ over analytic spaces and for Borel system dynamics:

1. bisimilar or behavioral equivalent stochastic right coalgebras are always logical equivalent,
2. If Λ is separating, then logical equivalent coalgebras are behavioral equivalent,
3. If Λ is separating and $\mathbb{F}$ has the Hennessy-Milner property, then behavioral equivalent coalgebras are bisimilar.

COROLLARY

For separating Λ and Hennessy-Milner functor $\mathbb{F}$,

$$\text{Logical equivalence} \Leftrightarrow \text{Bisimilarity} \Leftrightarrow \text{Behavioral equivalence.}$$

The Hennessy-Milner Property

The property

If $(S, k) \xrightarrow{f} (T, \ell) \xleftarrow{g} (S', k')$ is a cospan of surjective morphisms, then there exists a system dynamics $m : Q \to (\mathbb{F} \circ \mathbb{S})(Q)$ on

$$Q := \{\langle s, s' \rangle \mid f(s) = g(s')\}$$

such that the projections $(S, k) \xleftarrow{\pi_1} (Q, m) \xrightarrow{\pi_2} (S', k')$ form a span.

Theorem

The identity has the Hennessy-Milner property, and the class of functors having this property is closed under countable products and countable coproducts.

Remark

The proof hinges upon the Himmelberg-van Vleck Selection Theorem from stochastic dynamic programming, on the Hahn-Banach Theorem, and on the Riesz Representation Theorem.

GENERALITY

The general JAP-Theorem seems to be the most general characterization of bisimilarity for coalgebras based on the subprobability functor.

EXTENSION

Extension to more expressive logics (disjunction, negation, μ- and ν-operators) by adding natural transformations.

HENNESSY-MILNER

The Hennessy-Milner-property needs further investigation (and probably a counter example).

RIGHT VS. LEFT

Similar results hold for stochastic left coalgebras

$$S \mapsto (\mathbb{S} \circ \mathbb{F})(S).$$