Bisimilarity, behavioral and logical equivalence for stochastic right coalgebras

Ernst-Erich Doberkat
University of Dortmund
ls10-www.cs.uni-dortmund.de
(Joint work with Christoph Schubert)

Novosibirsk
We will discuss the logic given through

$$\varphi ::= \top | \varphi_1 \land \varphi_2 | \langle a \rangle \varphi$$

with $a \in \text{Act}$, $\text{Act}$ a countable alphabet of actions.

**Interpretation through Kripke models**

If $\mathcal{M} = (S, (\rightarrow_a)_{a \in \text{Act}})$ is a Kripke model (a.k.a. labeled transition system) over state space $S$, then

$$\mathcal{M}, s \models \langle a \rangle \varphi \text{ iff } \exists s' \in \mathcal{M} : s \rightarrow_a s'$$

(here $\mathcal{M} : s \models \varphi$).

**Theory of a state**

The theory of state $s \in S$ is as usual the set of formulas which are valid in this state,

$$Th_{\mathcal{M}}(s) := \{\varphi | \mathcal{M}, s \models \varphi\}.$$
Bisimilarity

Milner’s Definition

$B \subseteq S \times S'$ is a bisimulation between the Kripke models $\mathcal{M}$ and $\mathcal{M}'$ iff for all $\langle s, s' \rangle \in B$

- If $s \xrightarrow{a} t$, then $\exists t' \in S' : s' \xrightarrow{a}' t'$ and $\langle t, t' \rangle \in B$.
- If $s' \xrightarrow{a}' t'$, then $\exists t \in S : s \xrightarrow{a} t$ and $\langle t, t' \rangle \in B$.

Aczel’s Theorem

$B$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ iff there exists a transition structure $(\rightarrow''_a)_{a \in \mathcal{A}}$ on $B$ such that

$$\mathcal{M} \leftarrow (B, (\rightarrow''_a)_{a \in \mathcal{A}}) \rightarrow \mathcal{M}'.$$

Jan Rutten’s survey paper on coalgebras, TCS, 2000
Overview

1. **Markov Transition Systems**

2. **JAP-Theorem**

3. **Coalgebraic Logic**

4. **General JAP-Theorem**

5. **Concluding Remarks**
**Logic**

Stochastic version adds a lower bound for the probability of satisfaction to the modal operators:

\[ \varphi ::= \top | \varphi_1 \land \varphi_2 | \langle a \rangle_q \varphi \]

with \( q \in \mathbb{Q} \) and \( a \in \text{Act} \).

**Intuition:** State \( s \) satisfies \( \langle a \rangle_q \varphi \), provided the transition from \( s \) upon action \( a \) leads to a state that satisfies \( \varphi \) with probability at least \( q \).

**Stochastic Kripke model**

\( \mathcal{K} = (S, (k_a)_{a \in \text{Act}}) \) with \( k_a(s)(C) \) as the probability that action \( a \) in state \( s \) leads to a state in the measurable set \( C \subseteq S \). Note that \( k_a(s)(S) \leq 1 \).

**Interpretation**

\( \mathcal{K}, s \models \langle a \rangle_q \varphi \) iff \( k_a(s)([\varphi]) \geq q \).
Morphisms

Let $\mathcal{K} = (S, (k_a)_{a \in \mathcal{A}})$ and $\mathcal{K}' = (S', (k'_a)_{a \in \mathcal{A}})$ be stochastic Kripke models.

Morphism

A measurable and surjective map $f : S \rightarrow S'$ is a morphism $\mathcal{K} \rightarrow \mathcal{K}'$ iff

$$k'_a(f(s))(B') = k_a(s)(f^{-1}[B'])$$

for every measurable subset $B' \subseteq S'$ and every $s \in S$ and each action $a \in \mathcal{A}$.

Remark

$\mathcal{K}, s \models \varphi \iff \mathcal{K}', f(s) \models \varphi$ for morphism $f : \mathcal{K} \rightarrow \mathcal{K}'$.

$\mathcal{K}$ and $\mathcal{K}'$ are

- bisimilar iff $\mathcal{K} \leftrightarrow \mathcal{K}'' \rightarrow \mathcal{K}'$ for some $\mathcal{K}''$,
- behavioral equivalent iff $\mathcal{K} \rightarrow \mathcal{K}'' \leftarrow \mathcal{K}'$ for some $\mathcal{K}''$,
- logical equivalent iff $\{Th_\mathcal{K}(s) \mid s \in S\} = \{Th_{\mathcal{K}'}(s') \mid s' \in S'\}$. 
**Theorem**

For stochastic Kripke models over analytic spaces with Borel measurable transition laws, these equivalences hold

Logical equivalence $\Leftrightarrow$ Bisimilarity $\Leftrightarrow$ Behavioral equivalence.

**Remarks**


**More General Principle?**
Coalgebraic Reformulation
Let’s see

Coalgebraic view

A Markov transition system \( \mathcal{K} = (S, (k_a)_{a \in \mathcal{A}}) \) can be understood as a map

\[
k : S \to \prod_{a \in \mathcal{A}} \mathbb{S}(S) = (F \circ \mathbb{S})(S)
\]

with \( \mathbb{S}(S) \) as the set of all subprobabilities on \( S \).
Thus \( \mathcal{K} \) is a coalgebra \((S, k)\) for the functor \( F \circ \mathbb{S} \).

|= in terms of \( k \)?

Spell out

\[
s \models (a)_q \varphi \iff k_a(s)([\varphi]) \geq q
\]

\[
\iff k_a(s) \in \{ \mu \in \mathbb{S}(S) \mid \mu([\varphi]) \geq q \}
\]

\[
\iff s \in k^{-1} \left[ \{ m \in (F \circ \mathbb{S})(S) \mid \pi_a(m)([\varphi]) \geq q \} \right]
\]

\[
\iff s \in (k^{-1} \circ \lambda^{a,q}_S)([\varphi])
\]

with \( \lambda^{a,q}_S(D) := \{ m \in (F \circ \mathbb{S})(S) \mid \pi_a(m)(D) \geq q \} \).
**Coalgebraic Logic**

**Predicate liftings**

**Structure of \( \lambda \)**

\[ \lambda^{a,q}_S : \left( \text{measurable sets in } S \right) \to \left( \text{measurable sets in } (F \circ S)(S) \right). \]

\( \lambda^{a,q} \) is a natural transformation, called a **predicate lifting** for \( F \circ S \).

Predicate liftings arose originally from the work of L. Moss, D. Pattinson, and L. Schröder.

**Coalgebraic Logic**

Formulas are defined through

\[ \varphi ::= T \mid \varphi_1 \land \varphi_2 \mid \langle \lambda \rangle \varphi \]

with \( \lambda \in \Lambda \), \( \Lambda \) a set of predicate liftings for \( F \circ S \).

**Stochastic right coalgebra**

If \( F \) is a functor on a suitable category of measurable spaces, then \( (S, k) \) is a **stochastic right coalgebra** for \( F \) iff \( k : S \to (F \circ S)(S) \) is a measurable map, the system dynamics.
**Morphisms**

A measurable map $f : S \to S'$ is a morphism $\mathcal{R} \to \mathcal{R}'$ for the stochastic right coalgebras $\mathcal{R}$ and $\mathcal{R}'$ iff

$$k' \circ f = (\mathbb{F} \circ \mathbb{S})(f) \circ k.$$

**|=**

Interpret $\langle \lambda \rangle \varphi$ in the stochastic right coalgebra $\mathcal{R} = (S, k)$ through

$$\mathcal{R}, s \models \langle \lambda \rangle \varphi \text{ iff } s \in (k^{-1} \circ \lambda_S)(\llbracket \varphi \rrbracket).$$

**Observation**

Since each $\lambda \in \Lambda$ is natural, we have for the morphism $f : \mathcal{R} \to \mathcal{R}'$

$$\mathcal{R}, s \models \varphi \iff \mathcal{R}', f(s) \models \varphi.$$
Separation issues

The set Λ is assumed to be separating (“there are enough liftings”). Formally:

**Separation property**

If \( \text{Th}_R(s) \neq \text{Th}_{R'}(s') \) for states \( s \) and \( s' \) in a right coalgebra \( R \) resp. \( R' \), then there exists a formula \( \varphi \) and a lifting \( \lambda \in \Lambda \) such that

- either \( R, s \models \langle \lambda \rangle \varphi \)
- or \( R', s' \models \langle \lambda \rangle \varphi \)

holds.

Technically, separation relates

- the equivalence relation \( \varrho \) defined by the set of formulas (through \( s_1 \varrho s_2 \) iff \( \forall \varphi : s_1 \models \varphi \iff s_1 \models \varphi \))
- to the kernel \( \ker ((\mathbb{F} \circ S)(\eta_\varrho)) \) of the image of its factor map \( \eta_\varrho \) under \( \mathbb{F} \circ S \).
The following holds for stochastic right coalgebras $\mathcal{R}$ and $\mathcal{R}'$ over analytic spaces and for Borel system dynamics:

1. bisimilar or behavioral equivalent stochastic right coalgebras are always logical equivalent,
2. If $\Lambda$ is separating, then logical equivalent coalgebras are behavioral equivalent,
3. If $\Lambda$ is separating and $\mathcal{F}$ has the Hennessy-Milner property, then behavioral equivalent coalgebras are bisimilar.

**Corollary**

For separating $\Lambda$ and Hennessy-Milner functor $\mathcal{F}$,

Logical equivalence $\iff$ Bisimilarity $\iff$ Behavioral equivalence.
**The Hennessy-Milner Property**

**The property**

If $(S, k) \xrightarrow{f} (T, \ell) \xleftarrow{g} (S', k')$ is a cospan of surjective morphisms, then there exists a system dynamics \( m : Q \rightarrow (\mathcal{F} \circ S)(Q) \) on

\[
Q := \{ \langle s, s' \rangle \mid f(s) = g(s') \}
\]

such that the projections $(S, k) \xleftarrow{\pi_1} (Q, m) \xrightarrow{\pi_2} (S', k')$ form a span.

**Theorem**

The identity has the Hennessy-Milner property, and the class of functors having this property is closed under countable products and countable coproducts.

**Remark**

The proof hinges upon the Himmelberg-van Vleck Selection Theorem from stochastic dynamic programming, on the Hahn-Banach Theorem, and on the Riesz Representation Theorem.
**CONCLUDING REMARKS**

**WELL, THEN**

---

**GENERALITY**

The general JAP-Theorem seems to be the most general characterization of
bisimilarity for coalgebras based on the subprobability functor.

---

**EXTENSION**

Extension to more expressive logics (disjunction, negation, \( \mu \)- and
\( \nu \)-operators) by adding natural transformations.

---

**HENNESSY-MILNER**

The Hennessy-Milner-property needs further investigation (and probably a
counter example).

---

**RIGHT VS. LEFT**

Similar results hold for stochastic left coalgebras

\[
S \mapsto (S \circ F)(S).
\]