Effectivity on Subsets and Continuous Functions in Computable T_0-spaces

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We investigate aspects of effectivity and computability on open, closed and quasi-compact sets as well as partial continuous functions in computable T_0-spaces. A computable T_0-space is a second countable T_0-spaces with a notation of a base whose domain is recursive and computable intersection on the base. As we do not suppose our space to be a Hausdorff space, we don't talk about compact subsets. A set is called quasi-compact, if every open cover has a finite subcover.

We use the framework of the representation approach, TTE, where continuity and computability on finite and infinite sequences of symbols are defined canonically and transferred to abstract sets by means of notations and representations.

Computable Analysis connects Computability/Computational Complexity with Analysis/Numerical Computation by combining concepts of approximation and of computation. During the last 70 years various mutually non-equivalent models of real number computation have been proposed (Chap. 9 in [8]). Among these models the representation approach (Type-2 Theory of Effectivity, TTE) proposed by Grzegorczyk and Lacombe [5, 6] seems to be particularly realistic, flexible and expressive. So far the study of computability on sets of points, sets (open, closed, compact) and continuous functions has developed mainly bottom-up, i.e., from the real numbers to Euclidean space and metric spaces [11, 2, 9, 8, 12, 1, 13]. But often generalizations to more general spaces are needed (locally compact Hausdorff spaces [3], non-metrizable spaces [10], second countable T_0-spaces [7, 4]). This work is a generalization of the concepts introduced in [8] for the Euclidean case. Whenever reasonable, we transfer a representation to computable T0-spaces and discuss its properties and their relations to each other.

We use the concept of multi-valued partial functions. For a partial multi-function $f :\subseteq A \rightrightarrows B$, $f(a)$ is interpreted as the set of all results which are "acceptable" on input $a \in A$. Any concrete computation will produce on input $a \in \mathrm{dom}(f)$ some element $b \in f(a)$, but usually there is no method to select a specific one.

For a representation $\delta :\subseteq \Sigma^\omega \to M$, if $\delta(p) = x$ then the point $x \in M$ can be identified by the "name" $p \in \Sigma^\omega$. We will have applications where a sequence $p \in \Sigma^\omega$ contains information about a point x which is sufficient for some computation, although p does not identify x. We arrive at the concept of *multi-representation* $\delta :\subseteq \Sigma^\omega \rightrightarrows M$. A multi-representation can be considered as a naming system for the points of a set M where each name can encode many points. It can be interpreted also as a naming system of an *attribute* on M.

For computable T_0-spaces the set of all quasi-compact subsets or the set of continuous partial functions might be too big to have a representation. We can, however, introduce sufficiently meaningful multi-representations of these sets.

We define and compare various (multi-)representations of closed and quasi-compact sets as well as three multi-representations (open-open, via realization and via pointwise continuity) of the set of the partial continuous functions in computable T_0-spaces.

Keywords: Computable Analysis, TTE, (Multi-)Representation of Open, Closed and Quasi-Compact Sets, (Multi-)Representation of Partial Continuous Functions

References

[1] Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305:43–76, 2003.

[2] Vasco Brattka and Klaus Weihrauch. Computability on subsets of Euclidean space I: Closed and compact subsets. *Theoretical Computer Science*, 219:65–93, 1999.

[3] Pieter Collins. Continuity and computability on reachable sets. *Theoretical Computer Science*, 341:162–195, 2005.

[4] Tanja Grubba and Klaus Weihrauch. A computable version of Dini's theorem for topological spaces. In Pinar Yolum, Tunga Güngör, Fikret Gürgen, and Can Özturan, editors, *Computer and Information Sciences - ISCIS 2005*, volume 3733 of *Lecture Notes in Computer Science*, pages 927–936, Berlin, 2005. Springer. 20th International Symposium, ISCIS, Istanbul, Turkey, October 2005.

[5] Andrzej Grzegorczyk. Computable functionals. *Fundamenta Mathematicae*, 42:168–202, 1955.

[6] Daniel Lacombe. Classes récursivement fermés et fonctions majorantes. *Comptes Rendus Académie des Sciences Paris*, 240:716–718, June 1955. Théorie des fonctions.

[7] Matthias Schröder. Effective metrization of regular spaces. In Ker-I Ko, Anil Nerode, Marian B. Pour-El, Klaus Weihrauch, and Jiří Wiedermann, editors, *Computability and Complexity in Analysis*, volume 235 of *Informatik Berichte*, pages 63–80. FernUniversität Hagen, August 1998. CCA Workshop, Brno, Czech Republic, August, 1998.

[8] Klaus Weihrauch. *Computable Analysis.* Springer, Berlin, 2000.

[9] Mariko Yasugi, Takakazu Mori, and Yoshiki Tsujii. Effective properties of sets and functions in metric spaces with computability structure. *Theoretical Computer Science*, 219:467–486, 1999.

[10] Ning Zhong and Klaus Weihrauch. Computability theory of generalized functions. *Journal of the Association for Computing Machinery*, 50(4):469–505, 2003.

[11] Qing Zhou. Computable real-valued functions on recursive open and closed subsets of Euclidean space. *Mathematical Logic Quarterly*, 42:379–409, 1996.

[12] Martin Ziegler. Computability on regular subsets of Euclidean space. *Mathematical Logic Quarterly*, 48(Suppl. 1):157–181, 2002.

[13] Martin Ziegler. Computable operators on regular sets. *Mathematical Logic Quarterly*, 50(4,5):392–404, 2004.