Definability in the Homomorphic Quasiorder of Countable Labeled Forests

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This work was motivated mainly by extending the difference hierarchy to the case of partitions (see [1, 2, 4]). Definability for finite labeled forests was proved in [3]. Countable labeled forests were introduced and studied in [5].

By a countable forest (or just a forest) we mean (at most) countable poset in which every lower cone is a chain and which does not contain infinite chains. A tree is a forest having the least element (the root). A countable (labeled) k-forest (or just a k-forest) is an object $(F; \leq, c)$ consisting of a countable forest $(F; \leq)$ and a labeling $c : F \to k$, where natural number k ($k \geq 2$) is identified with the set $\{0, ..., k-1\}$. By natural numbers we denote also singletons (one element forests) labeled with corresponding numbers. Usually we simplify the notation $(F; \leq, c)$ to $(F; c)$ or even F. A morphism $f : (F_1; \leq_1, c_1) \to (F_2; \leq_2, c_2)$ between k-forests is a monotone function $f : (F_1; \leq_1) \to (F_2; \leq_2)$ respecting the labelings, i.e. satisfying $c_1 = c_2 \circ f$. Let $\widetilde{\mathcal{F}}_k$, $\widetilde{\mathcal{T}}_k$ and $\widetilde{\mathcal{T}}_k^i$ be the classes of all countable k-forests, all countable k-trees, and all countable k-trees with the roots labeled with i, respectively. As in [1, 2], we define a quasiorder $\leq$ on $\widetilde{\mathcal{F}}_k$ as follows: $F_1 \leq F_2$, if there is a morphism from F_1 to F_2. By $\equiv$ we denote the equivalence relation on $\widetilde{\mathcal{F}}_k$ induced by $\leq$.

The main result is the following:

Theorem 1 *For any $k \geq 3$, each element of the quotient structure of $(\widetilde{\mathcal{F}}_k; \leq, 0, ..., k-1)$ is definable in the language $L_{\omega_1\omega}$. The same is true for the quotient structures $(\widetilde{\mathcal{T}}_k; \leq, 0, ..., k-1)$ and $(\widetilde{\mathcal{T}}_k^i; \leq, i1, ..., i(k-1))$ $(i \in k)$.*

The function $X \to X'$ plays an important role in the proof, where for any $X \in \widetilde{\mathcal{T}}_k$ X' (modulo homomorphic equivalence) is the biggest element $Y \in \widetilde{\mathcal{F}}_k$ such that $Y < X$ (this element always exists). For inductive construction of a defining formula in the proof of the theorem, the following lemma is used:

Lemma 2 *If at least one of elements $X, Y \in \widetilde{\mathcal{T}}_k$ contains at least 3 different labels then $X' \equiv Y'$ implies $X \equiv Y$.*

Explicit computation of the function $X \to X'$ for countable k-trees used in the prove of this lemma is considerably different from the case of finite trees.

References

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