Bidomains and Non-deterministic Computation: Abstract

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In this talk, I will give an overview of an attempt to give quite a flexible account of sequential, non-deterministic higher-order computation, inspired by Berry's biorders, which may be presented of as posets with a binary operation (meet) and a coherence relation ("stable coherence"). We have shown that the stable and continuous functions over bidomains with $\top$ elements are all sequential, and so obtained definability results for (e.g.) the λ-calculus with meet operator. One possible interpretation of non-deterministic choice is as a meet ("demonic nondeterminism"), but there are others: *join* ("angelic"), *convex* '(combining the two) and as a weghted *probabilistic* sum, which we can model in essentially the same way.

Thus we define the notion of a generalized bicpo: a cpo $(D, \sqsubseteq)$ with a "coherence" $\uparrow$ (a symmetric, reflexive relation obeying an additional condition w.r.t. directed suprema) and a binary operation $* : D \times D \to D$ which is strict, continuous, idempotent, preserves coherence and satisfies the rule $((a * b) * (c * d)) = ((a * c) * (b * d))$. We obtain two categories in which objects are generalized bicpos: one in which maps are continuous functions which preserve coherence and the operation $*$ (multiplicativity or linearity) and one in which maps are continuous functions which preserve coherence and the product of coherent pairs — i.e. $f(a * b) = f(a) * f(b)$ if $a \uparrow b$ (conditional multiplicativity). This contains Berry's bicpos and stable and monotone functions as a full subcategory. A key result is that it is Cartesian closed.

Thus we may model functional languages with a choice operator as generalized bicpos and continuous and conditionally multiplicative functions. In particular, we may give CPS models (for e.g. PCF with first-class continuations) by fixing an appropriate totally ordered "answer object"' $\mathcal{R}$ on which all elements are coherent, and defining the action of $*$ on it. This gives the following denotational semantics:

- Setting $\mathcal{R} = \{\bot, \top\}$ and $* = \vee$ gives a model of erratic choice with respect to may-testing
- $\mathcal{R} = \{\bot, \top\}$, $* = \wedge$ gives a model of erratic choice with respect to must-testing
- Setting $\mathcal{R}$ to be the three-valued domain ("never", "sometimes", "always"), with $a*b = a$ if $a = b$ and $a * b =$ sometimes gives a model adequate for may-and-must testing.
- Setting $\mathcal{R} = [0, 1]$ and $*$ to be the weighted probabilistic choice $x * y = p.x + (1 - p).y$ for $p \in (0, 1)$ gives a model of the associated weighted choice.

Of these models, the first three are fully abstract and have the property that all compact elements are definable. The probabilistic model does not possess finite definability, and can separate the denotations of any λ-terms which are not $\beta\eta$-equivalent, for example. We do not know if it is fully abstract.

There are connections to other models of non-deterministic computation, including bistructures, probabilistic event structures and probabilistic interpretations of ludics. We have focussed on the relationship with non-deterministic game semantics: By adding a $\top$ move to a simple category of games or sequential algorithms, we may obtain a partial order and notion of coherence on deterministic strategies for each game, and by applying any of the four powerdomain constructions (Smyth, Hoare, Convex, Probabilistic) to this poset, giving symmetric monoidal closed categories of games and non-deterministic strategies in each case enriched over bidomains of the appropriate form. We show that each of these categories embeds fully in the category of generalised bidomains and linear functions.

In the meet and join cases we also have a full embedding of CCCs of ordered concrete data structures and non-deterministic sequential algorithms into the CCC of generalised bidomains and conditionally multiplicative functions, and in the probabilistic case, an embedding from a CCC of "history-sensitive" probabilistic games. We do not know whether this embedding is full or not.

Possibilities for further development include:

- By replacing the equality in the conditional multiplicativity requirement with inequality (lax generalised bidomains) we may model mixed choice.
- By replacing the operation $*$ with one of infinite arity we may model countable choice. In the meet case this requires us to relax continuity, but may still obtain full abstraction, the details are described elsewhere.