

# A computability flavoured view on Sobolev spaces

Atsushi Yoshikawa  
Kyushu University  
Fukuoka, 812-8581 Japan  
yoshikaw@math.kyushu-u.ac.jp

## Abstract

The Sobolev space is one of the most basic ideas in the present day mathematical analysis. It is defined as the Banach space, denoted, e.g.  $W_p^m(\Omega)$ , of Sobolev functions, i.e., measurable functions on a given domain  $\Omega$  in a Euclid space  $\mathbb{R}^n$ , having  $p$ -summable weak derivatives up to a given order  $m$ . It is introduced and thoroughly developed by Sergei Lebovich Sobolev, after whom this Institute is named.

In my talk, I begin by reviewing discussions in computable analysis on the Sobolev spaces themselves as have actually been carried out by many authors, including Marian B. Pour-El and Ning Zhong. The Sobolev space admits quite a comfortable formulation in computable analysis as far as the domain  $\Omega$  is *computably nice*. Then many problems classically formulated in it have a natural interpretation in computable analysis. However, not all the properties of the Sobolev space are computably valid even with  $\Omega$  having a computable boundary  $\partial\Omega$ .

The main part of my talk is my essay on discussing how *computably natural* the finite element method is. Namely, I discuss the computable nature of the piecewise polynomial approximation of Sobolev functions. In passing, I review Sobolev's classical projection of Sobolev functions in  $W_p^m(\Omega)$  onto its polynomial function subspace  $\mathbb{P}^{m-1}$  (of degree  $m-1$ ). This projectin, quite explicitly constructed, and it is shown that an estimate, now sometimes called the Bramble-Hilbert Lemma, is actually implicitly contained in Sobolev's original discussion. I also compare the Sobolev-Bramble-Hilbert projection with the orthogonal projection from  $W_2^m(\Omega)$  onto its subspace  $\mathbb{P}^{m-1}$  and show how different as well as how close they are. These observations are classically interesting, but serves also as a basic ingredients for computable analytic discussions.

Then I consider the (computable) domain  $\Omega$  and a class of families of (computable) subdomains  $\Omega_i^h$ ,  $i = 1, \dots, N_h$ , with the characteristic size  $h > 0$  of  $\Omega$  such that  $\bigcup_{i=1}^{N_h} \Omega_i^h$  approaches to  $\bar{\Omega}$  as  $h \downarrow 0$  and that, when  $\bar{\Omega}_i^h \cap \bar{\Omega}_j^h \neq \emptyset$ ,  $i \neq j$ , (i.e.,  $\Omega_i^h$  and  $\Omega_j^h$  are adjacent),  $\bar{\Omega}_i^h \cap \bar{\Omega}_j^h = \partial\Omega_i^h \cap \partial\Omega_j^h$  are hypersurfaces. Relevance of the parameter  $h$  in computable analysis is expected in the subsequent discussion. Then compare the Sobolev space  $W_p^m(\Omega)$  and the direct sum  $\bigoplus_{i=1}^{N_h} W_p^m(\Omega_i^h)$ . For  $u \in W_p^m(\Omega)$ , its restrictions  $u_i^h$  on  $\Omega_i^h$  belong to  $W_p^m(\Omega_i^h)$ , and the projections  $q_i^h \in \mathbb{P}^{m-1}$  of  $u_i^h$

in total realizes a piecewise polynomial approximation of the original  $u$  (in  $W_p^{m-1}(\Omega)$ , though).

The reverse procedure is exploited in the finite element method. Each  $\Omega_i^h$  corresponds to *an element domain*, and the subspace  $\mathbb{P}^{m-1}$  to the space generated by *shape functions*. For a certain given problem, one constructs each approximate polynomial solution  $q_i^h$  from the approximated problem on the subspace  $\mathbb{P}^{m-1}$  of  $W_p^m(\Omega_i^h)$ . These  $q_i^h$  are supposed to be compatible with the adjacent one, due to the choice of *nodal variables* which eventually restricts the configurations of  $\Omega_i^h$ . Then, the interpolant of these  $q_i^h$  is thus expected to give a piecewise polynomial approximation of the original problem. It is expected that these piecewise polynomial approximations approach the solution of the original problem as  $h \downarrow 0$ , and the limiting procedure is computable if everything is computable. I show that this in some cases this is in fact true, but yet to be worked out in general.

In my talk, I will not discuss any explicit problem from PDE, for instance. Thus, I do not discuss how each  $q_i^h$  is obtained. In particular, I will not consider any closed subspace of  $W_p^m(\Omega)$  specified by the boundary conditions although I need the boundary behaviors of Sobolev functions. The question is to analyze computably the meaning of the  $q_i^h \in \mathbb{P}^{m-1} \cap W_p^m(\Omega_i^h)$  which are compatible when adjacent.

So my purpose in the talk is to check and verify the relevance in computable analysis of the procedure of the finite element method.